

SIGMA-TERM PHYSICS

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February 2000

Pacs: 11.30.Rd, 12.39.Fe, 13.75.Gx, 13.75.Jz, 13.75.Lb

Keywords: chiral symmetries, chiral lagrangians, pion-baryon interactions,
kaon-baryon interactions, meson-meson interactions

Abstract

We consider sigma-terms in $\pi\pi$, πN and KN scattering. While the state of the art in $\pi\pi$ is in principle crystal clear, the kaon-nucleon case is largely unexplored. The pion-nucleon system is in-between. We propose a list of topics to be investigated in kaon-nucleon scattering, in order to make optimal use of the precision data expected from DEAR.

[†]Invited talk given at the Workshop *Physics and Detectors for DAΦNE*, Frascati, Nov. 16-19, 1999. To appear in the Proceedings.

1 Introduction

The sigma-terms considered here are proportional to the following matrix elements of scalar quark currents in the framework of QCD,

$$\langle A | m_q \bar{q}q | A \rangle \quad ; \quad q = u, d, s \quad ; \quad A = \pi, K, N .$$

These matrix elements are of interest, because they are related

- to the mass spectrum,
- to scattering amplitudes through Ward identities,
- to the strangeness content of A ,
- to quark mass ratios.

One may consider sigma-terms and the strangeness content e.g. in

$$\left. \begin{array}{l} \pi\pi \rightarrow \pi\pi \\ \pi N \rightarrow \pi N \\ KN \rightarrow KN \\ \bar{K}N \rightarrow \bar{K}N \\ \pi K \rightarrow \pi K \end{array} \right\} \quad \begin{array}{l} \text{where nearly everything is known,} \\ \text{much is known,} \\ \text{little is known.} \end{array}$$

In the following, we first illustrate the method to determine the sigma-term in the $\pi\pi$ sector, where plenty of information is available:

- from chiral perturbation theory [1, 2](ChPT): the amplitude to one loop off-shell [2] and to two loops on-shell [3],
- the σ -term to two loops [4],
- the scattering amplitude from data and from Roy equations [5].

Then, we briefly discuss the status in πN and compare these processes with kaon-nucleon scattering. We work in the isospin symmetry limit $m_u = m_d$ throughout.

2 Sigma-term in $\pi\pi$ scattering

We consider the elastic scattering process

$$\pi^-(q)\pi^0(p) \rightarrow \pi^-(q')\pi^0(p') \tag{2.1}$$

in QCD. We further introduce the standard Mandelstam variables $s = (p + q)^2, t = (q' - q)^2, u = (p' - q)^2$, use $\nu = s - u$ and denote the amplitude for the process (2.1) by $A(t, \nu)$.

2.1 The low-energy theorem for $\pi\pi$ scattering

The low-energy theorem for the on-shell amplitude reads

$$F_\pi^2 A(t, \nu) = \Gamma_\pi(t) + q'^\mu q^\nu r_{\mu\nu} \ , \quad (2.2)$$

where Γ_π denotes the scalar form factor of the pion,

$$\Gamma_\pi(t) = \langle \pi^0(p') | \hat{m}(\bar{u}u + \bar{d}d) | \pi^0(p) \rangle \ ; \ \hat{m} = \frac{1}{2}(m_u + m_d) \ . \quad (2.3)$$

The quantity $r_{\mu\nu}$ is not specified - the content of the theorem is the statement that $r_{\mu\nu}$ has the same analytic properties as the scattering amplitude itself - i.e., factoring the momenta q, q' in the manner shown in (2.2) does not introduce kinematic singularities in $r_{\mu\nu}$. Finally, the sigma-term is given by

$$2M_\pi \sigma_\pi = \Gamma_\pi(0) \ . \quad (2.4)$$

Therefore, in order to determine the sigma-term in this case, we need to measure the scattering amplitude, calculate the remainder $r_{\mu\nu}$, determine from this the scalar form factor $\Gamma_\pi(t)$ using (2.2), and evaluate it at $t = 0$. As is well known since the early days in sigma-term physics, one has to make sure that the low-energy theorem (2.2) is used in a kinematic region where the remainder $r_{\mu\nu}$ is small - otherwise, one introduces large uncertainties in the determination of the sigma-term. To illustrate, we note that the relation (2.2) is true order by order in ChPT. At leading order, the expressions read

$$F_\pi^2 A^{tree} = t - M_\pi^2 \ , \ \Gamma_\pi^{tree} = M_\pi^2 \ , \ r_{\mu\nu}^{tree} = -2g_{\mu\nu} \ . \quad (2.5)$$

Evaluating (2.2) at threshold, one finds that the remainder is twice as big (and of opposite sign) as the contribution from the scalar form factor itself, whereas the scalar form factor is entirely given by the scattering amplitude at $t = 2M_\pi^2$,

$$\begin{aligned} F_\pi^2 A^{tree} &= \Gamma_\pi^{tree} + \Delta_\pi^{tree} \ , \\ -1 &= 1 - 2 \quad \text{at threshold} \ , \\ 1 &= 1 + 0 \quad \text{at } t = 2M_\pi^2 \ , \end{aligned} \quad (2.6)$$

where the numbers are given in pion mass units. Therefore, it is not a good idea to use the low-energy theorem at threshold, although it is perfectly valid there, of course. Beyond tree level, the advantageous region - where the remainder is small - shrinks to the Cheng-Dashen (CD) point

$$t = 2M_\pi^2 \ , \ \nu = 0 \quad \text{Cheng-Dashen point} \ . \quad (2.7)$$

In the following, we write the relation (2.2) - evaluated at the CD point - in the form

$$F_\pi^2 A^{CD} = \Gamma_\pi(2M_\pi^2) + \Delta_\pi^{CD} . \quad (2.8)$$

The chiral expansion neatly illustrates the advantage of this choice. At the CD point, all quantities involved may be expanded in powers of the pion mass. For the amplitude itself, this expansion reads

$$A^{CD} = \frac{M_\pi^2}{F_\pi^2} - \frac{13M_\pi^4}{96\pi^2 F_\pi^4} \log \frac{M_\pi^2}{\Lambda^2} + O(M_\pi^6) , \quad (2.9)$$

where the scale Λ - which is independent of the pion mass - is related to the low-energy constants l_i in the chiral lagrangian [2],

$$\Lambda = 1.3 \text{ GeV} . \quad (2.10)$$

The logarithmic term in (2.9) is an example of the infrared singularities that show up in the chiral expansion [2]. The main point is that the scalar form factor contains an analogous singularity - they however nearly cancel in the difference Δ_π^{CD} ,

$$\begin{aligned} \Gamma_\pi(2M_\pi^2) &= M_\pi^2 - \frac{3M_\pi^4}{32\pi^2 F_\pi^2} \log \frac{M_\pi^2}{\Lambda_1^2} + O(M_\pi^6) , \\ \Delta_\pi^{CD} &= -\frac{M_\pi^4}{24\pi^2 F_\pi^2} \log \frac{M_\pi^2}{\Lambda_2^2} + O(M_\pi^6) , \\ \Lambda_1 &= 1.1 \text{ GeV} , \quad \Lambda_2 = 1.8 \text{ GeV} . \end{aligned} \quad (2.11)$$

Numerically, the relation (2.8) becomes at one-loop accuracy

$$\begin{aligned} 1.14 &= 1.09 + 0.05 \\ F_\pi^2 A^{CD} &= \Gamma_\pi(2M_\pi^2) + \Delta_\pi^{CD} , \end{aligned} \quad (2.12)$$

where the numbers are again in pion mass units. The remainder Δ_π^{CD} amounts to a correction of $0.05M_\pi/2 \simeq 3.5 \text{ MeV}$ - the same size as in the pion-nucleon case, see below.

2.2 The sigma-term from data

The above analysis and the relation (2.8) indicate how the sigma-term can be determined:

One evaluates

- A^{CD} from data on $\pi\pi \rightarrow \pi\pi$, using Roy equations [5].
- Dispersion relations for the form factor allow one to determine

$$\Delta_\sigma = \Gamma_\pi(2M_\pi^2) - \Gamma_\pi(0) . \quad (2.13)$$

- The quantity Δ_π^{CD} is determined from ChPT.

Then, one has

$$2M_\pi\sigma_\pi = F_\pi^2 A^{CD} - \Delta_\pi^{CD} - \Delta_\sigma . \quad (2.14)$$

It would be instructive to carry out this program with the available information on the $\pi\pi$ amplitude. Finally, we comment on the strangeness content of the pion,

$$y_\pi = \frac{2\langle\pi^0|\bar{s}s|\pi^0\rangle}{\langle\pi^0|\bar{u}u + \bar{d}d|\pi^0\rangle} . \quad (2.15)$$

From the expressions for the meson masses to one loop ChPT [2], one finds that y_π is of the order of a few percent. It is so small, because it is chirally suppressed, $y_\pi = O(M_\pi^2)$.

For illustration, we note that - at one-loop accuracy [2] - $\sigma_\pi \sim 69$ MeV.

3 Sigma-term in πN scattering

The analysis goes through in an analogous manner. The low-energy theorem reads in this case [6]

$$F_\pi^2 \bar{D}_{\pi N}^{CD} = \sigma_{\pi N}(2M_\pi^2) + \Delta_{\pi N}^{CD} , \quad (3.1)$$

where $\bar{D}_{\pi N}^{CD}$ denotes the isospin even D -amplitude of pion-nucleon scattering with pseudo-vector Born term subtracted. The quantity $\sigma_{\pi N}(t)$ is called the “scalar form factor of the proton”,

$$\bar{u}'u\sigma_{\pi N}(t) = \langle p'|\hat{m}(\bar{u}u + \bar{d}d)|p\rangle ; t = (p' - p)^2 , \quad (3.2)$$

where $|p\rangle$ denotes a one-proton state. The sigma-term and the strangeness content of the proton are

$$\sigma_{\pi N} = \sigma_{\pi N}(0) , y_N = \frac{2\langle p|\bar{s}s|p\rangle}{\langle p|\bar{u}u + \bar{d}d|p\rangle} . \quad (3.3)$$

The main point is that one can again use ChPT to determine $\Delta_{\pi N}^{CD}$: Whereas both the amplitude and the scalar form factor contain strong infrared singularities [7, 8, 9], these cancel in the present case completely up to and including terms of order p^4 [7], $\Delta_{\pi N}^{CD} = cM_\pi^4 + O(M_\pi^5)$, where the constant c is quark mass independent. Numerically, the correction is very small [7],

$$\Delta_{\pi N}^{CD} \sim 2 \text{ MeV} . \quad (3.4)$$

There are two coherent phase shift analyses available, KH80 and KA85. These then lead to a coherent analysis of the sigma-term [10],

$$\begin{aligned} \sigma_{\pi N} &\simeq 45 \text{ MeV} , \quad y_N \simeq 0.2 , \\ \sigma_{\pi N}(2M_\pi^2) - \sigma_{\pi N}(0) &\simeq 15 \text{ MeV} . \end{aligned} \quad (3.5)$$

This value for the sigma-term has recently been confirmed - using a different approach - by Büttiker and Meißner [11].

The phase shifts from KH80 and KA85 are essentially based on data acquired in the 70's - new data are included in the VPI/GW partial wave analyses. The most recent version is SP00 extending to 2.1 GeV above which KA84 amplitudes are employed. We plan to report [12] on the impact of e.g. SP00 in an analysis similar to the one performed in Ref. [10].

4 Kaon-nucleon sigma-terms

The investigation of the kaon-nucleon channels is much more involved, because

- there are less data,
- there are open channels below threshold,
- there is a resonance just below threshold in $\bar{K}N \rightarrow \bar{K}N$.

We refer the reader to Olin's contribution [13] for a detailed discussion of these issues, see also [14].

There are two σ -terms in this case [15], and we denote them by

$$\begin{aligned} \sigma_{KN}^u &= \frac{\hat{m} + m_s}{4m_p} \langle p | \bar{u}u + \bar{s}s | p \rangle , \\ \sigma_{KN}^d &= \frac{\hat{m} + m_s}{4m_p} \langle p | \bar{d}d + \bar{s}s | p \rangle , \end{aligned} \quad (4.1)$$

where $|p\rangle$ again denotes a one-proton state, and where m_p stands for the proton mass.

Although much work has already been performed [14, 15, 16], we believe that this is a field where many things remain to be done - we present some of the topics in the following agenda.

Agenda

- Establish the analogue of the low-energy theorem (2.2), without the use of formal manipulations with T -products, e.g. by working with the generating functional [17, 8]. Of course, $SU(3)$ breaking effects must be taken into account. In the following, we take it that the relation is

$$F_K^2 D_{KN}(t, \nu) = \sigma_{KN}^u(t) + q'_\mu q_\nu r_{KN}^{\mu\nu} \ , \quad (4.2)$$

where D_{KN} denotes the crossing symmetric amplitude, and where the remainder $r_{KN}^{\mu\nu}$ again has the same singularity structure as the amplitude itself.

- At the CD point $t = 2M_K^2, \nu = 0$, it then follows that

$$F_K^2 D_{KN}^{CD} = \sigma_{KN}^u(2M_K^2) + \Delta_{KN}^{CD} \ . \quad (4.3)$$

- evaluate D_{KN}^{CD} from data, e.g., relate it to scattering lengths [16]. Prepare to make use of the precise data that are expected from the DEAR [18] experiment. Note that the distance from threshold to the CD point is considerably larger here than in $\pi\pi$ or πN scattering.
- evaluate the remainder Δ_{KN}^{CD} in ChPT. Determine its infrared singularities. Check whether the correction is small compared to σ_{KN}^u .
- evaluate $\sigma_{KN}^u(2M_K^2) - \sigma_{KN}^u(0)$ from dispersion relations.
- Make use of chiral symmetry and ChPT whenever possible.

Why is all this interesting? One of the reasons is the fact that the sigma-term σ_{KN}^u can be related to the pion-nucleon sigma-term plus a remainder that is expected to be small,

$$\sigma_{KN}^u = (1 + y_N) \frac{\hat{m} + m_s}{4\hat{m}} \sigma_{\pi N} + \sigma_{KN}^{I=1} \ , \quad (4.4)$$

where

$$\sigma_{KN}^{I=1} = \frac{\hat{m} + m_s}{8m_p} \langle p | \bar{u}u - \bar{d}d | p \rangle \ , \quad (4.5)$$

and where isospin refers to the t -channel. The isospin zero part is

$$\sigma_{KN}^{I=0} = \frac{\hat{m} + m_s}{8m_p} \langle p | \bar{u}u + \bar{d}d + 2\bar{s}s | p \rangle . \quad (4.6)$$

The isospin one part is expected to be small,

$$\sigma_{KN}^{I=1} \sim \frac{m_s + \hat{m}}{m_s - \hat{m}} \frac{m_\Xi^2 - m_\Sigma^2}{8m_p} \sim 50 \text{ MeV} . \quad (4.7)$$

From Eq. (4.4), we conclude that one can determine y_N , by measuring σ_{KN}^u , $\sigma_{\pi N}$ and by estimating the isovector part $\sigma_{KN}^{I=1}$.

5 Off-shell methods

There are also Ward identities that relate the off-shell amplitudes to the sigma-term directly. In case of the $\pi\pi$ scattering amplitude, the relevant relation is

$$F_\pi^2 A(t, \nu; q'^2, q^2) = M_\pi^{-2} (q'^2 + q^2 - M_\pi^2) \Gamma_\pi(t) + q'^\mu q^\nu \bar{r}_{\mu\nu} , \quad (5.1)$$

where the off-shell amplitude is the one obtained by using the divergence of the axial current as the interpolating field for the pion. From this equation follow several exact relations, like

$$\begin{aligned} A(0, 0; 0, 0) &= -2M_\pi F_\pi^{-2} \sigma_\pi , \\ A(M_\pi^2, 0; 0, M_\pi^2) &= 0 , \\ A(M_\pi^2, 0; M_\pi^2, 0) &= 0 . \end{aligned} \quad (5.2)$$

The last two relations display the Adler zeros of the amplitude. One may now try to relate the off-shell point $t = \nu = q'^2 = q^2 = 0$ to the on-shell amplitude in the physical region, by use of the Adler zeros as constraints on the interpolation procedure. While the idea is beautiful, the beauty has its price: It is very difficult to control the approximations made in this extrapolation.

6 Summary and conclusions

1. Data, ChPT and dispersion relations allow one to pin down

$\sigma_{\pi\pi}$	very precisely,
$\sigma_{\pi N}$	less precisely,
$\sigma_{KN}^{u,d}$	barely until now.

2. DEAR may improve the situation as far as the kaon sigma-term is concerned, provided that one succeeds to relate this quantity in a reliable manner to the scattering lengths, such that the ones determined by the DEAR experiment [18] enter in a dominant manner.
3. This issue is a challenge for theoretical physicists, in particular, for the chiral symmetry framework.
4. As an intermediate step, it would be instructive to perform the analysis for $\pi K \rightarrow \pi K$ [19].

Acknowledgements

It is a pleasure to thank the organizers for the very informative week that we have spent in Frascati. This work has been supported in part by the Swiss National Science Foundation, and by TMR, BBW-Contract No. 97.0131 and EC-Contract No. ERBFMRX-CT980169 (EURODAΦNE).

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